LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – MATHEMATICS

THIRD SEMESTER – NOVEMBER 2009

MT 3810 / 3803 / 3800 - TOPOLOGY

Date & Time: 03/11/2009 / 9:00 - 12:00 Dept. No. Max. : 100 Marks

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1) a) i) Let X be a metric space with metric d .Show that $d_1(x,y)$ defined by $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$, is also a metric on X .

OR

ii) Let C(X, ¡) be the set of all bounded continuous real functions defined on the metric space X and B be the set of all bounded real functions defined on X. Prove that C(X, ¡) is a closed subset of the metric space B . (5)

b) i) Let X be a metric space. Prove that any arbitrary union of open sets in X is open and any finite intersection of open sets in X is open.

ii) Give an example to show that any arbitrary intersection of open sets in X is not open.

iii) In any metric space X , show that a subset F of X is closed $\Leftrightarrow$ its complement F' is open. (6+3+6)

OR

iv) Let X be a complete metric space, and let Y be a subspace of X . Show that Y is complete $\Leftrightarrow$ it is closed.

v) Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Prove that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

vi) State and prove Baire's Theorem. (6+5+4)

2) a) i) Prove that every separable metric space is second countable.

OR

ii) Let X be any non-empty set, and let S be an arbitrary class of subsets of X . Prove that the class of all unions of finite intersection of sets in S is a topology. (5)

b) i) Show that any closed subspace of a compact space is compact.

ii) Give an example to show that a proper subspace of a compact space need not be closed.

iii) Prove that any continuous image of a compact space is compact. (6+3+6)

OR

iv) Let $C(X, ¡)$ be the set of all bounded continuous real functions defined on a topological space X . Show that $C(X, ¡)$ is a real Banach space with respect to pointuise addition and scalar multiplication and the norm defined by $\|f\| = \sup|f(x)|$; (2) if multiplication is defined pointuise, $C(X, ¡)$ is a commutative real algebra with identity in which $\|fg\| \le \|f\|\|g\|$ *and* $\|1\| = 1$. (15)

3) a) i) Prove that a metric space is sequentially compact $\Leftrightarrow$ it has the Bolzano Weierstrass property.

OR

ii) Show that a closed subspace of a complete metric space is compact $\Leftrightarrow$ it is totally bounded. (5)

b) i) State and prove Lebesque's covering Lemma.

ii) Show that every sequentially compact metric space is compact. (10+5)

OR

iii) Prove that the product of any non-empty class of compact spaces is compact.

iv) Show that any continuous mapping of a compact metric space into a compact metric space is uniformly continuous. (6+9)

4) a) i) Prove that every compact Hausdorff space is normal.

OR

ii) In a Hausdorff space, show that any point and disjoint compact subspace can be separated by open sets. (5)

b) i) State and prove the Tietze Extension Theorem.

OR

ii) If X is a second countable normal space, show that there exists a homeomorphism f of X onto a subspace of ¡ $^{¥}$, and X is therefore metrizelle. (15)

5) a) i) Show that any continuous image of a connected space is connected.

OR

ii) Let X be a compact Hausdorff space. Show that X is totally disconnected $\Leftrightarrow$ it has an open base whose sets are also closed. (5)

b) i) Let X be a topological space and A be a connected subspace of X. If B is a subspace of X such that $A \subseteq B \subseteq \bar{A}$, then show that B is connected.

ii) If X is an arbitrary topological space, then prove the following:
1) each point in X is contained in exactly one component of X;
2) each connected subspace of X is contained in a component of X;
3) a connected subspace of X which is both open and closed is a component of X.
(3+12)

OR

iii) Let f be a continuous real function defined on a closed interval [a,b], and let $\in > 0$ be given. Prove that there exists a polynomial p with real coefficients such that $|f(x) - p(x)| < \in$ for all x in $[a,b]$ (15)

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